



5.8 Cauchy-Bunyakovsky-Schwarz inequality

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Find an upper bound for the absolute value of the inner product
 - Give and prove the Cauchy-Bunyakovsky-Schwarz inequality
 - Look at some examples





Inequalities

- We have seen a number of inequalities:

$$|w + z| \leq |w| + |z|$$

$$\|\mathbf{u} + \mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$





Cauchy-Bunyakovsky-Schwarz inequality

- We can also bound the magnitude of the inner product:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$

– For example, if $\|\mathbf{u}\|_2 = 3$ and $\|\mathbf{v}\|_2 = 4$, then $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq 12$

– It can be zero, for if $\mathbf{u}$ and $\mathbf{v}$ are non-zero and orthogonal, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 0 < \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$





Cauchy-Bunyakovsky-Schwarz inequality

Theorem

For all vectors, $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$

Proof:

If $\mathbf{u} = \mathbf{0}$, then $|\langle \mathbf{0}, \mathbf{v} \rangle| = |0| = 0 = 0 \cdot \|\mathbf{v}\|_2 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$
thus the inequality is true.

Assume $\mathbf{u} \neq \mathbf{0}$, so

$$\mathbf{v} = \text{proj}_{\mathbf{u}}(\mathbf{v}) + \text{perp}_{\mathbf{u}}(\mathbf{v})$$

Because the projection and perpendicular components are perpendicular, we may apply the Pythagorean theorem

$$\|\mathbf{v}\|_2^2 = \|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 + \|\text{perp}_{\mathbf{u}}(\mathbf{v})\|_2^2$$





Cauchy-Bunyakovsky-Schwarz inequality

Now, all three values are non-negative, so if

$$\|\mathbf{v}\|_2^2 = \|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 + \|\text{perp}_{\mathbf{u}}(\mathbf{v})\|_2^2$$

then

$$\|\mathbf{v}\|_2^2 \geq \|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2$$

$$\text{Thus, } \|\mathbf{v}\|_2^2 \geq \left\| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \right\|_2^2$$

$$\text{but } \|\alpha \mathbf{u}\|_2 = |\alpha| \|\mathbf{u}\|_2 \text{ so } \|\alpha \mathbf{u}\|_2^2 = |\alpha|^2 \|\mathbf{u}\|_2^2$$

$$\text{Applying this, we have } \|\mathbf{v}\|_2^2 \geq \left\| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \right\|_2^2 = \left| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \right|^2 \|\mathbf{u}\|_2^2$$





Cauchy-Bunyakovsky-Schwarz inequality

Using the properties of the absolute value:

$$\begin{aligned}\|\mathbf{v}\|_2^2 &\geq \left| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \right|^2 \|\mathbf{u}\|_2^2 \\ &= \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\cancel{\langle \mathbf{u}, \mathbf{u} \rangle^2}} \cancel{\|\mathbf{u}\|_2^2} \\ &= \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{u}\|_2^2}\end{aligned}$$

But $\langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|_2^2$

Thus, $\|\mathbf{v}\|_2^2 \geq \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{u}\|_2^2}$, so $\|\mathbf{u}\|_2^2 \|\mathbf{v}\|_2^2 \geq |\langle \mathbf{u}, \mathbf{v} \rangle|^2$.

The square root function is monotonic, so therefore

$$\|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \geq |\langle \mathbf{u}, \mathbf{v} \rangle| \quad \blacksquare$$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3 \cdot 12 + 4 \cdot 5 = 56$$

$$\|\mathbf{u}\|_2 = \sqrt{3^2 + 4^2} = 5$$

$$\|\mathbf{v}\|_2 = \sqrt{12^2 + 5^2} = 13$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 56 \leq 65 = 5 \cdot 13 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2 \cdot (-3) + 1 \cdot 2 + 2 \cdot (-6) = -16$$

$$\|\mathbf{u}\|_2 = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\|\mathbf{v}\|_2 = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 16 \leq 21 = 3 \cdot 7 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 6 \\ -9 \end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot (-3) + (-2) \cdot 6 + 3 \cdot (-9) = -42$$

$$\|\mathbf{u}\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\|_2 = \sqrt{3^2 + 6^2 + 9^2} = \sqrt{126}$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 42 \leq 42 = \sqrt{14} \cdot \sqrt{126} = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 0.3 \\ -0.5 \\ 0.1 \\ 1.2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0.2 \\ -2.4 \\ -0.6 \\ -1 \end{pmatrix}$

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= 0.3 \cdot 0.2 + (-0.5) \cdot (-2.4) + 0.1 \cdot (-0.6) + 1.2 \cdot (-1) \\ &= 0.06 + 1.2 - 0.06 - 1.2 = 0\end{aligned}$$

$$\|\mathbf{u}\|_2 = \sqrt{0.3^2 + 0.5^2 + 0.1^2 + 1.2^2} = \sqrt{1.79}$$

$$\|\mathbf{v}\|_2 = \sqrt{0.2^2 + 2.4^2 + 0.6^2 + 1^2} = \sqrt{7.16}$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 0 \leq 3.58 = \sqrt{1.79} \cdot \sqrt{7.16} = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$





Summary

- Following this topic, you now
 - Know the Cauchy-Bunyakovsky-Schwarz inequality bounds the absolute value of the inner product
 - Seen the proof
 - Looked at a few examples





References

- [1] https://en.wikipedia.org/wiki/Cauchy-Bunyakovsky-Schwarz_inequality
- [2] https://en.wikipedia.org/wiki/Inner_product_space





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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